

CLASSIFICATION OF BI-QUTRIT PPT ENTANGLED EDGE STATES BY THEIR RANKS

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ABSTRACT. We construct $3 \otimes 3$ PPT entangled edge states with maximal ranks, to complete the classification of $3 \otimes 3$ PPT entangled edge states by their types. The ranks of the states and their partial transposes are 8 and 6, respectively. These examples also disprove claims in the literature.

1. INTRODUCTION

Let M_n denote the C^* -algebra of all $n \times n$ matrices over the complex field, with the cone M_n^+ of all positive semi-definite matrices. A positive semi-definite matrix in $M_m \otimes M_n$ is said to be separable if it is the convex sum of rank one projectors onto product vectors $x \otimes y \in \mathbb{C}^m \otimes \mathbb{C}^n$. A positive semi-definite matrix in $M_m \otimes M_n$ is said to be entangled if it is not separable. Since the convex cone of all separable ones coincides with $M_m^+ \otimes M_n^+$, the entanglement consists of $(M_m \otimes M_n)^+ \setminus M_m^+ \otimes M_n^+$. The notion of entanglement is a unique phenomenon in non-commutative order structures, and there is no counterpart in classical mechanics. Indeed, it is well-known that the equality $(\mathcal{A} \otimes \mathcal{B})^+ = \mathcal{A}^+ \otimes \mathcal{B}^+$ holds for commutative C^* -algebras \mathcal{A} and \mathcal{B} which are mathematical frameworks for classical mechanics. This notion of quantum entanglement has been one of the key research topics since the nineties, in relation with possible applications to quantum information theory and quantum computation theory.

One of the main research topics in the theory of entanglement is to distinguish entanglement from separability. If we take a rank one projector onto a product vector $x \otimes y$, then it is easy to see that its partial transpose is also a rank one projector onto the product vector $\bar{x} \otimes y$, where \bar{x} denotes the vector whose entries are complex conjugates of the corresponding entries of the vector $x \in \mathbb{C}^m$. Recall that the partial transpose $(X \otimes Y)^\tau$ is given by $X^t \otimes Y$ with the usual transpose X^t of X . Therefore, if $A \in M_m \otimes M_n$ is separable, then its partial transpose A^τ is also positive semi-definite, as was observed by Choi [9] and Peres [25]. A block matrix $A \in M_m \otimes M_n$ is said to be of PPT (positive partial transpose) if both of A and A^τ are positive semi-definite. The notion of PPT turns out to be very important in quantum physics in relation with bound entanglement. See [16].

Woronowicz [33] showed that if $m = 2$ and $n \leq 3$ then the notions of separability and PPT coincide, and gave an explicit example of entanglement $A \in M_2 \otimes M_4$ which is of PPT. This

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kind of block matrix is called a PPTES (positive partial transpose entangled state) when it is normalized. The first example of PPTES in $M_3 \otimes M_3$ was found by Choi [9]. A PPTES A is said to be a PPT entangled edge state, or just an edge state in short, if there exists no nonzero product vector $x \otimes y \in \mathcal{R}A$ with $\bar{x} \otimes y \in \mathcal{R}A^\tau$ as was introduced in [23], where $\mathcal{R}A$ denotes the range space of A . In other words, edge states violate the range criterion for separability [18] in an extreme way.

Since every PPT state is the convex sum of a separable state and an edge state, it is essential to classify edge states to understand the whole structures of PPT states. The first step to classify them is to use the ranks. A PPT state A is said to be of type (p, q) if the rank of A is p and the rank of A^τ is q , as was introduced in [26]. Now, we concentrate on the case of $3 \otimes 3$. By the results in [19, 20], we have the following possibilities of types for $3 \otimes 3$ PPT entangled edge states:

$$(1) \quad (4, 4), (5, 5), (6, 5), (7, 5), (8, 5), (6, 6), (7, 6), (8, 6),$$

here we list up types (p, q) with $p \geq q$ by the symmetry. See [3, 9, 10, 11, 13, 14, 18, 31] for concrete examples of $3 \otimes 3$ edge states of various types. We refer to [20] for a summary of examples. All possibilities have been realized in the literature mentioned above, except for the case of $(8, 6)$. In fact, it has been claimed in [26] that if there is a $3 \otimes 3$ PPT entangled edge state of type (p, q) then $p + q \leq 13$.

The purpose of this note is to present two parameterized examples of $3 \otimes 3$ PPT entangled edge states of type $(8, 6)$, to complete the classification of $3 \otimes 3$ edges states by their types. These examples disprove the above mentioned claim in [26]. Our examples also disprove another claim [22] that if $D = (\mathcal{R}A)^\perp$ and $E = (\mathcal{R}A^\tau)^\perp$ for a PPT state $A \in M_m \otimes M_n$ and $\dim D + \dim E = m + n - 2$, then there exist finitely many product vectors $x \otimes y \in \mathcal{R}A$ with $\bar{x} \otimes y \in \mathcal{R}A^\tau$.

After we explain in the next section the notion of PPT edge states in the context of the whole convex structures of the convex cone generated by PPT states, we present our construction of two parameterized examples of edges of type $(8, 6)$ in the Section 3. In the last section, we also exhibit various types of edge states arising from this construction.

2. CONVEX GEOMETRY OF PPT STATES

We denote by \mathbb{V}_1 and \mathbb{T} the convex cones generated by all separable and PPT states, respectively. The PPT criterion by Choi and Peres tells us that the relation $\mathbb{V}_1 \subset \mathbb{T}$ holds. One of the best way to understand the whole structures of a given convex set is to characterize the lattice of all faces. We have very few general information for the facial structures of the convex cone \mathbb{V}_1 itself. See [1] in this direction. On the other hand, we have an easy way to describe faces of the cone \mathbb{T} generated by PPT states.

Every faces of the cone \mathbb{T} is determined [12] by a pair of subspaces of $\mathbb{C}^m \otimes \mathbb{C}^n$. More precisely, every face of \mathbb{T} is of the form

$$\sigma(D, E) = \{A \in \mathbb{T} : \mathcal{R}A \subseteq D, \mathcal{R}A^\tau \subseteq E\}$$

for a pair (D, E) of subspaces of $\mathbb{C}^m \otimes \mathbb{C}^n$. Nevertheless, it is very difficult in general to determine which pairs of subspaces give rise to faces of the convex cone \mathbb{T} , and this difficulty is one of the main motivation of this note. In the case of $2 \otimes 2$, all faces of \mathbb{T} have been found [12] in terms of pairs of subspaces, using the facial structures [4, 30] of the convex cone of all positive linear maps between M_2 . Recall that a point x of a convex set C is said to be an interior point of C if the line segment from any point of C to x may be extended within C . The set of all interior point of C will be denoted by $\text{int } C$, which is nothing but the relative interior of C with respect to the affine manifold generated by C . Note that $\text{int } C$ is never empty for any convex set C . A point of C which is not an interior point is said to be a boundary point. The set of all boundary points of C will be denoted by ∂C . We recall that the interior of $\sigma(D, E)$ is given by

$$\text{int } \sigma(D, E) = \{A \in \mathbb{T} : \mathcal{R}A = D, \mathcal{R}A^\tau = E\}.$$

From now on, we compare boundary structures of the two convex cones \mathbb{V}_1 and \mathbb{T} . Basically, we have the following four cases for a given face $\sigma(D, E)$ of the cone \mathbb{T} :

- $\sigma(D, E) \subseteq \mathbb{V}_1$.
- $\sigma(D, E) \not\subseteq \mathbb{V}_1$ but $\text{int } \sigma(D, E) \cap \mathbb{V}_1 \neq \emptyset$.
- $\text{int } \sigma(D, E) \cap \mathbb{V}_1 = \emptyset$ but $\partial \sigma(D, E) \cap \mathbb{V}_1 \neq \emptyset$.
- $\sigma(D, E) \cap \mathbb{V}_1 = \emptyset$.

Recall that the range criterion for separability tells us that if a PPT state A is separable with $D = \mathcal{R}A$ and $E = \mathcal{R}A^\tau$ then there exist product vectors $x_i \otimes y_i \in \mathbb{C}^m \otimes \mathbb{C}^n$ such that

$$D = \text{span} \{x_i \otimes y_i\}, \quad E = \text{span} \{\bar{x}_i \otimes y_i\}.$$

We say that a pair (D, E) satisfies the range criterion if there exist product vectors with the above property. Therefore, we see that if the interior of $\sigma(D, E)$ has a nonempty intersection with the cone \mathbb{V}_1 then (D, E) satisfies the range criterion. The converse of this statement is also true as was shown in [7], even though the converse of the range criterion itself does not hold. In short, we see that (D, E) satisfies the range criterion if and only if the first two conditions among the above four hold. In terms of a PPT state A itself, we see that $(\mathcal{R}A, \mathcal{R}A^\tau)$ satisfies the range criterion if and only if the smallest face containing A has a separable state in its interior. Recall that every point x of a convex set determines a unique face in which x is an interior point. This is the smallest face containing x .

It remains two cases to be considered: A face $\sigma(D, E)$ either touches the cone \mathbb{V}_1 at the boundary or never touches the cone \mathbb{V}_1 . It is easy to see that the latter case occurs if and only if every element of the face $\sigma(D, E)$ is a PPT entangled edge state. If this is the case with $\dim D = p$ and $\dim E = q$ then every interior point of the face $\sigma(D, E)$ is an edge state of type (p, q) , and every boundary point of $\sigma(D, E)$ is also an edge state of type (s, t) with $s < p$ or $t < q$.

The first step to characterize the lattice of all faces of the cone \mathbb{T} is to find all pairs (p, q) of natural numbers for which there exists a face $\sigma(D, E)$ with $\dim D = p$ and $\dim E = q$. See [22] for this line of research. This classification is especially important for the cases of separable

states and edge states, since every PPT state is the sum of a separable state and an edge state. This task for separable states is nothing but to classify the dimensions of pairs of subspaces satisfying the range criterion. In the case $2 \otimes n$, all pairs (p, q) of natural numbers have been characterized [7] for which there exist pairs (D, E) satisfying the range criterion with $\dim D = p$ and $\dim E = q$.

As for edge states, there are previous results in the literature in two directions. It was shown [19, 24] that if the rank of $A \in M_m \otimes M_n$ is less than or equal to $\max\{m, n\}$ then two notions of PPT and separability coincide. This gives a lower bound for the ranks of A and A^τ for an edge state $A \in M_m \otimes M_n$: If A is an $m \otimes n$ edge of type (p, q) , then we have

$$p, q > \max\{m, n\}.$$

On the other hand, for a given pair (D, E) of subspaces in $\mathbb{C}^m \otimes \mathbb{C}^n$, it was shown in [20] that there must exist $x \otimes y \in D$ with $\bar{x} \otimes y \in E$, whenever either the inequality

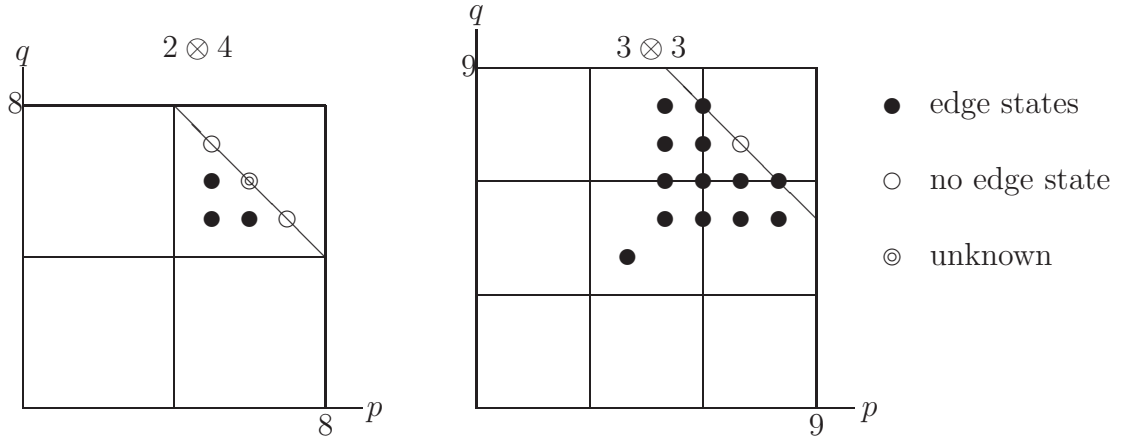
$$\dim D + \dim E > 2mn - m - n + 2$$

holds, or $\dim D + \dim E = 2mn - m - n + 2$ and

$$\sum_{r+s=m-1} (-1)^r \binom{k}{r} \binom{\ell}{s} \neq 0$$

hold with $k = \dim D^\perp$ and $\ell = \dim E^\perp$. This gives us an upper bound for the ranks of A and A^τ for an edge state $A \in M_m \otimes M_n$.

In case of $m = n = 3$, we have $2mn - m - n + 2 = 14$. It is easy to see that $(k, \ell) = (2, 2)$ satisfies the above condition, but $(k, \ell) = (1, 3)$ does not satisfy. Furthermore, it is now known [5, 28] that every PPT entanglement of rank 4 is automatically of type $(4, 4)$. All of these arguments give us the possibilities of types as is given in (1). See also [20] for the summary in the case of $(m, n) = (2, 4)$ as well as in the case of $m = n = 3$. It is unknown whether there exists a $2 \otimes 4$ PPT edge state of type $(6, 6)$ or not. Classifications of possible types of edge states for the $2 \otimes 4$ and $3 \otimes 3$ cases are summarized in the following pictures:



3. CONSTRUCTION

We begin with the following 3×3 matrix

$$P[\theta] := \begin{pmatrix} e^{i\theta} + e^{-i\theta} & -e^{i\theta} & -e^{-i\theta} \\ -e^{-i\theta} & e^{i\theta} + e^{-i\theta} & -e^{i\theta} \\ -e^{i\theta} & -e^{-i\theta} & e^{i\theta} + e^{-i\theta} \end{pmatrix}$$

which has a kernel vector $(1, 1, 1)^t$. Considering the principal submatrices, we see that $P[\theta]$ is positive semi-definite if and only if $\cos \theta \geq 0$ and $2 \cos 2\theta \geq -1$ if and only if $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$. If $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$ then $P[\theta]$ is of rank two, and if $\theta = -\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$ then $P[\theta]$ is of rank one.

Consider the following matrix

$$(2) \quad A = \begin{pmatrix} e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & \frac{1}{b} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} \end{pmatrix}$$

in $M_3 \otimes M_3$ with the conditions

$$(3) \quad b > 0, \quad -\frac{\pi}{3} < \theta < \frac{\pi}{3}, \quad \theta \neq 0,$$

where \cdot denote zero. The partial transpose A^τ of A is given by

$$A^\tau = \begin{pmatrix} e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot \\ \cdot & -e^{i\theta} & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot \\ \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} \end{pmatrix}.$$

It is clear that A is of PPT under the condition (3), and we have $\text{rank } A = 8$ and $\text{rank } A^\tau = 6$.

We proceed to show that A is a PPT entangled edge state under the condition (3). First of all, we note that the kernel of A is spanned by

$$(1, 0, 0; 0, 1, 0; 0, 0, 1)^t$$

and the kernel of A^τ is spanned by the following three vectors:

$$\begin{aligned} &(0, b, 0; e^{i\theta}, 0, 0; 0, 0, 0)^t, \\ &(0, 0, 0; 0, 0, b; 0, e^{i\theta}, 0)^t, \\ &(0, 0, e^{i\theta}; 0, 0, 0; b, 0, 0)^t. \end{aligned}$$

Suppose that a product vector $z = x \otimes y \in \mathbb{C}^3 \otimes \mathbb{C}^3$ is in the range of A , and $\bar{x} \otimes y$ is in the range of A^τ . Then we have

$$(4) \quad x_1 y_1 + x_2 y_2 + x_3 y_3 = 0,$$

and

$$(5) \quad \begin{aligned} b \bar{x}_1 y_2 + e^{-i\theta} \bar{x}_2 y_1 &= 0, \\ b \bar{x}_2 y_3 + e^{-i\theta} \bar{x}_3 y_2 &= 0, \\ b \bar{x}_3 y_1 + e^{-i\theta} \bar{x}_1 y_3 &= 0. \end{aligned}$$

From (5) we see that at least one of x_i, y_i is zero. Indeed, we have

$$b^3 \bar{x}_1 \bar{x}_2 \bar{x}_3 y_1 y_2 y_3 = -e^{-3i\theta} \bar{x}_1 \bar{x}_2 \bar{x}_3 y_1 y_2 y_3$$

by (5), from which $\bar{x}_1 \bar{x}_2 \bar{x}_3 y_1 y_2 y_3 = 0$. If $x \otimes y$ is nonzero, then we also have $x_i = 0 \iff y_i = 0$ from (5).

We first consider the case of $x_3 = y_3 = 0$. Then we have

$$x_1 y_1 + x_2 y_2 = 0, \quad b \bar{x}_1 y_2 + e^{-i\theta} \bar{x}_2 y_1 = 0,$$

from which we have

$$\bar{x}_1 x_1 y_1 + \bar{x}_1 x_2 y_2 = 0, \quad b \bar{x}_1 x_2 y_2 + e^{-i\theta} \bar{x}_2 x_2 y_1 = 0.$$

Therefore, we get

$$|x_1|^2 y_1 = -\bar{x}_1 x_2 y_2 = \frac{e^{-i\theta}}{b} |x_2|^2 y_1.$$

Since $\theta \neq 0$, we conclude that $x_1 = x_2 = 0$ or $y_1 = 0$. If $x_1 = x_2 = 0$, then $x = 0$. If $y_1 = 0$ and either x_1 or x_2 is nonzero, then we have $y = 0$. Similar arguments for the cases $x_1 = y_1 = 0$ and $x_2 = y_2 = 0$ show that if $x, y \in \mathbb{C}^3$ satisfy the relations (4) and (5), then $x \otimes y = 0$. This shows that there exists no nonzero product vector $x \otimes y \in \mathcal{R}A$ with $\bar{x} \otimes y \in \mathcal{R}A^\tau$. Therefore, we conclude that A is a PPT entangled edge state of type (8, 6).

Recall that every 5-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ has a product vector. See [32]. This is equivalent to say that every system of equations consisting of four homogeneous linear equations with respect to unknowns $\{x_i y_j : i, j = 1, 2, 3\}$ must have nontrivial solutions. But, the system of four equations from (4) and (5) involve complex conjugates, and may not have nonzero solutions. This seems to be the main point for the wrong statements in [26] and [22].

For nonnegative real numbers a, b and c , we consider the following linear map

$$\Phi[a, b, c](X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix}$$

between M_3 , as was introduced in [6]. We also recall that the Choi matrix $C_\phi \in M_m \otimes M_n$ of a linear map $\phi : M_m \rightarrow M_n$ is given by

$$C_\phi := \sum_{i,j=1}^m e_{ij} \otimes \phi(e_{ij}) \in M_m \otimes M_n,$$

and C_ϕ is of PPT if and only if ϕ is both completely positive and completely copositive by [8]. We also note that $\Phi[a, b, c]$ is both completely positive and completely copositive if and only if $a \geq 2$ and $bc \geq 1$ by [6]. If $\theta = 0$ then the matrix A in (2) is just the Choi matrix of the map $\Phi[2, b, \frac{1}{b}]$, which is a PPT state of type $(8, 6)$. On the other hand, we have the following PPT states

$$(6) \quad A = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{b} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

of type $(7, 6)$ in the literature [13], which is an edge state whenever $b > 0$ and $b \neq 1$. The key idea of the construction was to parameterized offdiagonals -1 and 1 of these two cases by $e^{i\theta}$. We note that a variant of (6) has been used by Størmer [31] to give a short proof of the indecomposability of the Choi map $\Phi[1, 0, \lambda]$ for $\lambda \geq 1$.

If $\theta = 0$ then A in (2) turns out to be separable. Indeed, if we take product vectors

$$\begin{aligned} z_1(\omega) &= (0, 1, \sqrt{b}\omega) \otimes (0, \sqrt{b}, -\bar{\omega}) = (0, 0, 0; 0, \sqrt{b}, -\bar{\omega}; 0, b\omega, -\sqrt{b}) \\ z_2(\omega) &= (\sqrt{b}\omega, 0, 1) \otimes (-\bar{\omega}, 0, \sqrt{b}) = (-\sqrt{b}, 0, b\omega; 0, 0, 0; -\bar{\omega}, 0, \sqrt{b}) \\ z_3(\omega) &= (1, \sqrt{b}\omega, 0) \otimes (\sqrt{b}, -\bar{\omega}, 0) = (\sqrt{b}, -\bar{\omega}, 0; b\omega, -\sqrt{b}, 0; 0, 0, 0) \end{aligned}$$

in $\mathbb{C}^3 \otimes \mathbb{C}^3$ then it is straightforward to see that

$$A = \frac{1}{3b} \sum_{i=1}^3 \sum_{\omega \in \Omega} z_i(\omega) z_i(\omega)^*,$$

where $\Omega = \{1, e^{\frac{2}{3}\pi i}, e^{-\frac{2}{3}\pi i}\}$ is the third roots of unity. Therefore, we see that the Choi matrix of the map $\Phi[a, b, c]$ is of PPT if and only if it is separable if and only if $a \geq 2$ and $bc \geq 1$. This shows that the linear map $\Phi[a, b, c]$ is super-positive in the sense of [2], or equivalently an entanglement breaking channel in the sense of [17] and [21] if and only if it is both completely positive and completely copositive if and only if $a \geq 2$ and $bc \geq 1$. See [27] for related topics.

If we put the following number

$$a_\theta = \max\{e^{i(\theta+\frac{3}{2}\pi)} + e^{-i(\theta+\frac{3}{2}\pi)}, e^{i\theta} + e^{-i\theta}, e^{i(\theta-\frac{3}{2}\pi)} + e^{-i(\theta-\frac{3}{2}\pi)}\}$$

in the place of $e^{i\theta} + e^{-i\theta}$ when we define the matrix A in (2), then we have similar PPT edge states for every θ . Note that a_θ is the smallest number so that

$$\begin{pmatrix} a_\theta & -e^{i\theta} & -e^{-i\theta} \\ -e^{-i\theta} & a_\theta & -e^{i\theta} \\ -e^{i\theta} & -e^{-i\theta} & a_\theta \end{pmatrix}$$

is positive semi-definite.

4. EDGE STATES OF OTHER TYPES

Let A be the matrix given by (2). Now, we search edge states X in the smallest face containing A by a similar method as in [13]. Note that X is in this face if and only if the relations

$$\mathcal{R}X \subseteq \mathcal{R}A, \quad \mathcal{R}X^\tau \subseteq \mathcal{R}A^\tau$$

hold. Note that every range vector of A^τ is of the form

$$\left(\xi_i, \frac{\alpha_i}{\sqrt{b}}, -\gamma_i \sqrt{b} e^{i\theta}; -\alpha_i \sqrt{b} e^{i\theta}, \eta_i, \frac{\beta_i}{\sqrt{b}}; \frac{\gamma_i}{\sqrt{b}}, -\beta_i \sqrt{b} e^{i\theta}, \zeta_i \right)^\tau,$$

for scalars $\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i$ and γ_i . We denote by P the rank one projector onto the vector

$$\left(1, \frac{1}{\sqrt{b}}, -\sqrt{b} e^{i\theta}; -\sqrt{b} e^{i\theta}, 1, \frac{1}{\sqrt{b}}; \frac{1}{\sqrt{b}}, -\sqrt{b} e^{i\theta}, 1 \right)^\tau,$$

and by Q_i onto the vector

$$(\xi_i, \alpha_i, \gamma_i; \alpha_i, \eta_i, \beta_i; \gamma_i, \beta_i, \zeta_i)^\tau,$$

respectively. Then we see that A^τ is the Hadamard product of P and $\sum_i Q_i$ for suitable choice of $\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i$ and γ_i . If we write $\xi, \eta, \zeta, \alpha, \beta$ and γ the vectors whose i -th entries are $\xi_i, \eta_i, \zeta_i, \alpha_i, \beta_i$ and γ_i , respectively, then the matrix $X = (X^\tau)^\tau$ is the Hadamard product of the following two matrices:

$$\begin{pmatrix} 1 & \frac{1}{\sqrt{b}} & -\sqrt{b} e^{-i\theta} & -\sqrt{b} e^{i\theta} & -e^{i\theta} & b & \frac{1}{\sqrt{b}} & \frac{1}{b} & -e^{-i\theta} \\ \frac{1}{\sqrt{b}} & \frac{1}{b} & -e^{-i\theta} & 1 & \frac{1}{\sqrt{b}} & -\sqrt{b} e^{-i\theta} & -\sqrt{b} e^{i\theta} & -e^{i\theta} & b \\ -\sqrt{b} e^{i\theta} & -e^{i\theta} & b & \frac{1}{\sqrt{b}} & \frac{1}{b} & -e^{-i\theta} & 1 & \frac{1}{\sqrt{b}} & -\sqrt{b} e^{-i\theta} \\ -\sqrt{b} e^{-i\theta} & 1 & \frac{1}{\sqrt{b}} & b & -\sqrt{b} e^{i\theta} & -e^{i\theta} & -e^{-i\theta} & \frac{1}{\sqrt{b}} & \frac{1}{b} \\ -e^{-i\theta} & \frac{1}{\sqrt{b}} & \frac{1}{b} & -\sqrt{b} e^{-i\theta} & 1 & \frac{1}{\sqrt{b}} & b & -\sqrt{b} e^{i\theta} & -e^{i\theta} \\ b & -\sqrt{b} e^{i\theta} & -e^{i\theta} & -e^{-i\theta} & \frac{1}{\sqrt{b}} & \frac{1}{b} & -\sqrt{b} e^{-i\theta} & 1 & \frac{1}{\sqrt{b}} \\ \frac{1}{\sqrt{b}} & -\sqrt{b} e^{-i\theta} & 1 & -e^{i\theta} & b & -\sqrt{b} e^{i\theta} & \frac{1}{b} & -e^{-i\theta} & \frac{1}{\sqrt{b}} \\ \frac{1}{b} & -e^{-i\theta} & \frac{1}{\sqrt{b}} & \frac{1}{b} & -\sqrt{b} e^{-i\theta} & 1 & -e^{i\theta} & b & -\sqrt{b} e^{i\theta} \\ -e^{i\theta} & b & -\sqrt{b} e^{i\theta} & \frac{1}{b} & -e^{-i\theta} & \frac{1}{\sqrt{b}} & \frac{1}{b} & -\sqrt{b} e^{-i\theta} & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} (\xi|\xi) & (\xi|\alpha) & (\xi|\gamma) & (\alpha|\xi) & (\alpha|\alpha) & (\alpha|\gamma) & (\gamma|\xi) & (\gamma|\alpha) & (\gamma|\gamma) \\ (\alpha|\xi) & (\alpha|\alpha) & (\alpha|\gamma) & (\eta|\xi) & (\eta|\alpha) & (\eta|\gamma) & (\beta|\xi) & (\beta|\alpha) & (\beta|\gamma) \\ (\gamma|\xi) & (\gamma|\alpha) & (\gamma|\gamma) & (\beta|\xi) & (\beta|\alpha) & (\beta|\gamma) & (\zeta|\xi) & (\zeta|\alpha) & (\zeta|\gamma) \\ (\xi|\alpha) & (\xi|\eta) & (\xi|\beta) & (\alpha|\alpha) & (\alpha|\eta) & (\alpha|\beta) & (\gamma|\alpha) & (\gamma|\eta) & (\gamma|\beta) \\ (\alpha|\alpha) & (\alpha|\eta) & (\alpha|\beta) & (\eta|\alpha) & (\eta|\eta) & (\eta|\beta) & (\beta|\alpha) & (\beta|\eta) & (\beta|\beta) \\ (\gamma|\alpha) & (\gamma|\eta) & (\gamma|\beta) & (\beta|\alpha) & (\beta|\eta) & (\beta|\beta) & (\zeta|\alpha) & (\zeta|\eta) & (\zeta|\beta) \\ (\xi|\gamma) & (\xi|\beta) & (\xi|\zeta) & (\alpha|\gamma) & (\alpha|\beta) & (\alpha|\zeta) & (\gamma|\gamma) & (\gamma|\beta) & (\gamma|\zeta) \\ (\alpha|\gamma) & (\alpha|\beta) & (\alpha|\zeta) & (\eta|\gamma) & (\eta|\beta) & (\eta|\zeta) & (\beta|\gamma) & (\beta|\beta) & (\beta|\zeta) \\ (\gamma|\gamma) & (\gamma|\beta) & (\gamma|\zeta) & (\beta|\gamma) & (\beta|\beta) & (\beta|\zeta) & (\zeta|\gamma) & (\zeta|\beta) & (\zeta|\zeta) \end{pmatrix}.$$

Since $(1, 0, 0; 0, 1, 0; 0, 0, 1)^t$ is in the $\text{Ker } A \subseteq \text{Ker } X$, we have

$$\|\xi\|^2 = e^{i\theta}\|\alpha\|^2 + e^{-i\theta}\|\gamma\|^2, \quad \|\eta\|^2 = e^{i\theta}\|\beta\|^2 + e^{-i\theta}\|\alpha\|^2, \quad \|\zeta\|^2 = e^{i\theta}\|\gamma\|^2 + e^{-i\theta}\|\beta\|^2,$$

and so we have $\|\alpha\| = \|\beta\| = \|\gamma\|$. If $\|\alpha\| = \|\beta\| = \|\gamma\| = 0$, then $A = 0$. So, we may assume that

$$\|\alpha\| = \|\beta\| = \|\gamma\| = 1.$$

Then we have

$$(7) \quad \|\xi\|^2 = \|\eta\|^2 = \|\zeta\|^2 = e^{i\theta} + e^{-i\theta}.$$

Considering $(2, 4)$, $(6, 8)$ and $(7, 3)$ principal submatrices, we also have

$$(8) \quad |(\xi|\eta)| \leq 1, \quad |(\eta|\zeta)| \leq 1, \quad |(\zeta|\xi)| \leq 1.$$

If we take vectors so that $\text{span}\{\xi, \eta, \zeta\} \perp \text{span}\{\alpha, \beta, \gamma\}$ with mutually orthonormal vectors α, β, γ then we have

$$X = \begin{pmatrix} e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & \frac{1}{b} & \cdot & (\eta|\xi) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & (\zeta|\xi) & \cdot & \cdot \\ \cdot & (\xi|\eta) & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & (\zeta|\eta) & \cdot \\ \cdot & \cdot & (\xi|\zeta) & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & (\eta|\zeta) & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} \end{pmatrix}$$

and

$$X^\tau = \begin{pmatrix} e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & (\xi|\eta) & \cdot & \cdot & \cdot & (\xi|\zeta) \\ \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot \\ \cdot & -e^{i\theta} & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ (\eta|\xi) & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & (\eta|\zeta) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot \\ \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & b & \cdot \\ (\zeta|\xi) & \cdot & \cdot & \cdot & (\zeta|\eta) & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} \end{pmatrix}.$$

It is clear that X is of PPT under the conditions (7) and (8). We note that the rank of X is equal to

$$2 + \text{rank} \begin{pmatrix} \frac{1}{b} & (\xi|\eta) \\ (\eta|\xi) & b \end{pmatrix} + \text{rank} \begin{pmatrix} \frac{1}{b} & (\eta|\zeta) \\ (\zeta|\eta) & b \end{pmatrix} + \text{rank} \begin{pmatrix} b & (\zeta|\xi) \\ (\xi|\zeta) & \frac{1}{b} \end{pmatrix}$$

and the rank of X^τ is equal to

$$3 + \text{rank} \begin{pmatrix} e^{i\theta} + e^{-i\theta} & (\xi|\eta) & (\xi|\zeta) \\ (\eta|\xi) & e^{i\theta} + e^{-i\theta} & (\eta|\zeta) \\ (\zeta|\xi) & (\zeta|\eta) & e^{i\theta} + e^{-i\theta} \end{pmatrix} = 3 + \dim \text{span} \{ \xi, \eta, \zeta \}.$$

In the three dimensional space \mathbb{C}^3 , it is possible to take linearly independent vectors ξ, η, ζ satisfying (7) and (8) so that some of

$$(\xi|\eta), \quad (\eta|\zeta), \quad (\zeta|\xi)$$

are of absolute values one and the remainders are zero. Therefore, we get examples of edge states of types (8, 6), (7, 6), (6, 6) and (5, 6).

To get edge states of type $(p, 5)$ for $p = 5, 6, 7, 8$, it is convenient to consider the matrix

$$P[\rho, \sigma, \tau] := \begin{pmatrix} e^{i\theta} + e^{-i\theta} & \rho & \bar{\tau} \\ \bar{\rho} & e^{i\theta} + e^{-i\theta} & \sigma \\ \tau & \bar{\sigma} & e^{i\theta} + e^{-i\theta} \end{pmatrix}$$

with the conditions

$$(9) \quad |\rho| \leq 1, \quad |\sigma| \leq 1, \quad |\tau| \leq 1, \quad \det P[\rho, \sigma, \tau] = 0.$$

Then $P[\rho, \sigma, \tau]$ is a positive semi-definite matrix of rank two. By spectral decomposition, we may get two vectors $E_1 = (\xi_1, \eta_1, \zeta_1)$ and $E_2 = (\xi_2, \eta_2, \zeta_2)$ so that $P[\rho, \sigma, \tau]$ is the sum of rank one projectors onto E_1 and E_2 , respectively. Then we see that

$$P[\rho, \sigma, \tau] = \begin{pmatrix} |\xi_1|^2 & \xi_1 \bar{\eta}_1 & \xi_1 \bar{\zeta}_1 \\ \eta_1 \bar{\xi}_1 & |\eta_1|^2 & \eta_1 \bar{\zeta}_1 \\ \zeta_1 \bar{\xi}_1 & \zeta_1 \bar{\eta}_1 & |\zeta_1|^2 \end{pmatrix} + \begin{pmatrix} |\xi_2|^2 & \xi_2 \bar{\eta}_2 & \xi_2 \bar{\zeta}_2 \\ \eta_2 \bar{\xi}_2 & |\eta_2|^2 & \eta_2 \bar{\zeta}_2 \\ \zeta_2 \bar{\xi}_2 & \zeta_2 \bar{\eta}_2 & |\zeta_2|^2 \end{pmatrix} = \begin{pmatrix} (\xi|\xi) & (\xi|\eta) & (\xi|\zeta) \\ (\eta|\xi) & (\eta|\eta) & (\eta|\zeta) \\ (\zeta|\xi) & (\zeta|\eta) & (\zeta|\zeta) \end{pmatrix}.$$

If we take ρ, σ, τ with (9) so that some of them are of absolute values one and the remainders of them have the absolute values less than one, then we get PPT entangled states of types (8, 5), (7, 5), (6, 5) and (5, 5).

For a given fixed θ with (3), we can take a real number r with $-1 < r < 1$ so that

$$P[r, r, r], \quad P[r, -r, 1], \quad P[1, 1, r]$$

is of rank two, respectively, to get edge states of types (8, 5), (7, 5) and (6, 5). For example, we see that

$$P[-\cos \theta, -\cos \theta, -\cos \theta]$$

is of rank two, and so we get the following natural examples of $3 \otimes 3$ edge states of type (8, 5):

$$\begin{pmatrix} e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & \frac{1}{b} & \cdot & -\cos \theta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & -\cos \theta & \cdot & \cdot \\ \cdot & -\cos \theta & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & -\cos \theta & \cdot \\ \cdot & \cdot & -\cos \theta & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -\cos \theta & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} \end{pmatrix}.$$

To get examples of edge states of types (7, 5) and (6, 5), we put $\omega = e^{i\theta} + e^{-i\theta}$ temporarily. Note that $1 < \omega < 2$. We also note that

$$\det P[r, -r, 1] = (1 + \omega)(\omega^2 - \omega - 2r^2), \quad \det P[1, 1, r] = (\omega - r)(r\omega + \omega^2 - 2).$$

and zeros

$$r = \sqrt{\frac{\omega^2 - \omega}{2}} = \sqrt{2 \cos^2 \theta - \cos \theta}, \quad r = \frac{2 - \omega^2}{\omega} = -\frac{\cos 2\theta}{\cos \theta}$$

of them are in the interval $(-1, 1)$, respectively. In this way, we get edge states of types (7, 5) and (6, 5). If we consider the rank two matrix $P[-e^{i\theta}, -e^{i\theta}, -e^{i\theta}]$, which is nothing but $P[\theta]$ at the beginning of the construction, then we have the following parameterized example of edge states of type (5, 5):

$$\begin{pmatrix} e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot \\ \cdot & -e^{i\theta} & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot \\ \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta} + e^{-i\theta} \end{pmatrix}.$$

In conclusion, we have constructed $3 \otimes 3$ PPT entangled edge states of type (8, 6) whose existence has been a long-standing question since the claim in [26] without proof. In this vein, it would be also an interesting question whether there exists a $2 \otimes 4$ edge states of type (6, 6) or not, as was explained in [20]. We have shown that there exist edge states of all possible types in the smallest faces of the the PPT states we constructed, except for edge states of (4, 4) types. These include parameterized examples of edge states of types (5, 5) and (6, 6), for which there have been known very few discrete examples [10, 11]. We also have natural parameterized examples of edge states of type (8, 5). Compare with [13]. We note that the study of bi-qutrit edge states with minimal ranks was initiated by [3], and have been recently studied in [5, 28, 15, 29] very extensively. It is the authors' hope that this is the starting point for the further study of bi-qutrit edge states with maximal ranks.

REFERENCES

- [1] E. Alfsen and F. Shultz, *Unique decompositions, faces, and automorphisms of separable states*, J. Math. Phys. **51**(2010), 052201.
- [2] T. Ando, *Cones and norms in the tensor product of matrix spaces*, Linear Algebra Appl. **379** (2004), 3–41.
- [3] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, *Unextendible product bases and bound entanglement*, Phys. Rev. Lett. **82** (1999), 5385–5388.
- [4] E.-S. Byeon and S.-H. Kye, *Facial structures for positive linear maps in the two dimensional matrix algebra*, Positivity **6** (2002), 369–380.
- [5] L. Chen and D. Ž. Djoković, *Description of rank four PPT entangled states of two qutrits*, J. Math. Phys. **52** (2011), 122203.
- [6] S.-J. Cho, S.-H. Kye, and S. G. Lee, *Generalized Choi maps in 3-dimensional matrix algebras*, Linear Alg. Appl. **171** (1992), 213–224.
- [7] H.-S. Choi and S.-H. Kye, *Facial structures for separable states*, J. Korean Math. Soc., **49** (2012), to appear, <http://www.math.snu.ac.kr/~kye/paper/separable.pdf>.
- [8] M.-D. Choi, *Completely positive linear maps on complex matrices*, Linear Alg. Appl. **10** (1975), 285–290.
- [9] M.-D. Choi, *Positive linear maps*, Operator Algebras and Applications (Kingston, 1980), pp. 583–590, Proc. Sympos. Pure Math. Vol 38. Part 2, Amer. Math. Soc., 1982.
- [10] L. Clarisse, *Construction of bound entangled edge states with special ranks*, Phys. Lett. A **359** (2006), 603–607.
- [11] K.-C. Ha, *Comment on : “Construction of bound entangled edge states with special ranks”* [Phys. Lett. A 359 (2006) 603], Phys. Lett. A **361** (2007), 515–519.
- [12] K.-C. Ha and S.-H. Kye, *Construction of entangled states with positive partial transposes based on indecomposable positive linear maps*, Phys. Lett. A **325** (2004), 315–323.
- [13] K.-C. Ha and S.-H. Kye, *Construction of $3 \otimes 3$ entangled edge states with positive partial transposes*, J. Phys. A **38** (2005), 9039–9050.
- [14] K.-C. Ha, S.-H. Kye, and Y. S. Park, *Entanglements with positive partial transposes arising from indecomposable positive linear maps*, Phys. Lett. A **313** (2003), 163–174.
- [15] L. O. Hansen, A. Hauge, J. Myrheim, and P. Ø. Sollid, *Low rank positive partial transpose states and their relation to product vectors*, preprint, arXiv:1104.1519.
- [16] M. Horodecki, P. Horodecki and R. Horodecki, *Mixed-state entanglement and distillation: is there a “bound” entanglement in nature?*, Phys. Rev. Lett. **80** (1998), 5239–5242.
- [17] M. Horodecki, P. W. Shor and M. B. Ruskai, *General entanglement breaking channels*, Rev. Math. Phys. **15** (2003), 629–641.
- [18] P. Horodecki, *Separability criterion and inseparable mixed states with positive partial transposition*, Phys. Lett. A **232** (1997), 333–339.
- [19] P. Horodecki, M. Lewenstein, G. Vidal, and I. Cirac, *Operational criterion and constructive checks for the separability of low rank density matrices*, Phys. Rev. A **62** (2000), 032310.
- [20] Y.-H. Kiem, S.-H. Kye, and J. Lee, *Existence of product vectors and their partial conjugates in a pair of spaces*, J. Math. Phys. **52** (2011), 122201.
- [21] A. S. Kholevo, M. E. Shirokov and R. F. Werner, *On the notion of entanglement in Hilbert spaces*, Russian Math. Surveys **60** (2005), 359–360.
- [22] J. M. Leinaas, J. Myrheim, and P. Ø. Sollid, *Numerical studies of entangled PPT states in composite quantum systems*, Phys. Rev. A **81** (2010), 062329.
- [23] M. Lewenstein, B. Kraus, P. Horodecki, and J. I. Cirac, *Characterization of separable states and entanglement witnesses*, Phys. Rev. A **63** (2001), 044304.
- [24] B. Kraus, J. I. Cirac, S. Karnas, and M. Lewenstein, *Separability in $2 \times N$ composite quantum systems*, Phys. Rev. A **61** (2000), 062302.
- [25] A. Peres, *Separability criterion for density matrices*, Phys. Rev. Lett. **77** (1996), 1413–1415.
- [26] A. Sanpera, D. Bruß, and M. Lewenstein, *Schmidt-number witnesses and bound entanglement*, Phys. Rev. A **63** (2001), 050301.
- [27] A. I. Singh, *Quantum Dynamical Semigroups involving Separable and Entangled States*, preprint, arXiv:1201.0250.
- [28] L. Skowronek, *Three-by-three bound entanglement with general unextendible product bases*, J. Math. Phys. **52** (2011), 122202.
- [29] P. Ø. Sollid, J. M. Leinaas, and J. Myrheim, *Unextendible product bases and extremal density matrices with positive partial transpose*, Phys. Rev. A **84** (2011), 042325.

- [30] E. Størmer, *Positive linear maps of operator algebras*, Acta Math. **110** (1963), 233–278.
- [31] E. Størmer, *Decomposable positive maps on C^* -algebras*, Proc. Amer. Math. Soc. **86** (1982), 402–404.
- [32] N. R. Wallach, *An Unentangled Gleason’s Theorem*, Contemp. Math. **305** (2002), 291–298.
- [33] S. L. Woronowicz, *Positive maps of low dimensional matrix algebras*, Rep. Math. Phys. **10** (1976), 165–183.

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